Vilniaus universitetas
Duomenų mokslo ir skaitmeninių technologijų institutas
L I E T U V A

INFORMATIKA (N009)

STOCHASTINIŲ DINAMINIŲ SISTEMŲ, STEBIMŲ SU TRIUKŠMU, FILTRAVIMO, IDENTIFIKAVIMO IR VALDYMO REALIU LAIKU ALGORITMŲ SUDARYMAS IR TAIKYMAS

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2020 m. spalis

Mokslinė ataskaita DMSTI-DS-N009-20-2

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Santrauka

Ataskaitoje sutrumpinta forma yra pateikiama 2019 - 2020 m. m. straipsniams rengta medžiaga. Ši medžiaga yra dvejopa: viena jos dalis yra skirta teoriniams tyrimams, kuriais siekiama sukurti atitinkamus algoritmus (pagal disertacijos tematiką), o kita dalis – pasirinktos šių algoritmų taikymo srities plėtojimui.

Raktiniai žodžiai: Gaussian random walk, maximum likelihood, recursion, online estimation, social capital, cultural processes, logistic regression, stochastic differential equation.

1 Įvadas

Šioje ataskaitoje yra glaustai pateikiami 2019 – 2020 m. m. vykdytų mokslinių tyrimų rezultatai.

Pirmoji darbo dalis (antrasis skyrius) yra skirta teorinei mokslinių tyrimų daliai. Joje sprendžiamas atsitiktinio Gauso klaidžiojimo, stebimo su triukšmu, parametrų vertinimo realiu laiku uždavinys. Uždavinio sprendimui yra sukonstruojamas didžiausio tikėtinumo metodu paremtas rekursinis algoritmas. Algoritmo veikimas ištiriamas eksperimentiškai.

Antroji darbo dalis (trečiasis skyrius) yra skirta praktiniam teorinių rezultatų taikymui. Joje yra pasiūlomas kultūros poveikio socialiniam kapitalui tikimybinis modelis, tokiu būdu siekiant kuriamus algoritmus su stochastinėmis dinaminės sistemomis susijusių uždavinių sprendimui pritaikyti socialinių sistemų simuliavimui.

Pažymėtina, kad antrosios dalies rezultatai yra atspausdinti WoS žurnale (*L. Sakalauskas, V. Dulskis, R. Lauzikas, A. Miliauskas, D. Plikynas (2020) A probabilistic model of the impact of cultural participation on social capital, The Journal of Mathematical Sociology, DOI: 10.1080/0022250X.2020.1725002*). Tuo tarpu visapusiškai papildytą pirmosios dalies turinį su pridėtais teoriniais rezultatais (šioje ataskaitoje ši dalis pateikiama nėra) planuojama pateikti spausdinimui iki šių metų pabaigos.

Atsižvelgiant į tai, kad antrojo bei trečiojo skyrių medžiagą sudaro straipsniams skirta medžiaga, jų turinys pateikiamas anglų kalba.

2 Incremental Maximum Likelihood Estimation of the Parameters of Noisy Gaussian Random Walk

2.1 Problem Formulation

Let us consider a probability measure space $(\Omega,\mathcal{F},P)$. Suppose $\{X_i\}$, $i \in \mathbb{Z}^+$, is a discrete-time linear stochastic state process, taking values in $\mathbb{R}$, with dynamics given by

$$X_{i+1} = X_i + \epsilon_{i+1}, \quad X_0 = 0. \tag{2.1}$$

Here $\{\epsilon_{i+1}\}$, $i \in \mathbb{Z}^+$, is a sequence of independent and identically distributed $\mathcal{N}(0,Q)$ scalar random variables ($\mathcal{N}(\mu,\sigma^2)$ denotes the normal distribution with mean μ and variance σ^2).

The state process $\{X_i\}$, $i \in \mathbb{Z}^+$, is a well-known Gaussian Random Walk (see, e.g., [1]), which, in turn, is a special case of the more general Linear Gaussian State Space Model [2]. It is observed indirectly via the scalar observation process $\{Y_i\}$, $i \in \mathbb{Z}^+$, given by

$$Y_i = X_i + \nu_i. \tag{2.2}$$

Here $\{\nu_i\}$, $i \in \mathbb{Z}^+$, is a sequence of independent and identically distributed $\mathcal{N}(0,R)$ scalar random variables. It is assumed that $\{\epsilon_{i+1}\}$ and $\{\nu_i\}$, $i \in \mathbb{Z}^+$, are mutually independent.

The model described by equations (2.1) and (2.2) is completely characterized by parameter $\theta := (Q,R)$. We consider the problem of incremental (online) maximum likelihood estimation of θ.

2.2 Solution

The goal of this chapter is to arrive at the algorithm for the problem formulated in the previous chapter. Firstly, we step-by-step introduce the necessary preliminaries while making comments about the algorithm along the way, and later we combine everything into the formal description of the algorithm.

2.2.1 Preliminaries

Let us define increments between adjacent observations as $Z_k := Y_k - Y_{k-1}$, $k = 1$, ..., n, $n \in \mathbb{N}$. The motivation for using differences Z instead of raw observations Y is that they produce a more compact covariance matrix, i.e. a tridiagonal one. We have

$$\mathrm{E}Z_k = 0$$

and

$$\mathrm{Cov}(Z_k,Z_l) = \mathrm{E}(Z_k Z_l) = \begin{cases} Q + 2R, & k = l \\ -R, & |k-l| = 1 \\ 0, & otherwise \end{cases},$$

here k, $l = 1$, ..., n, $n \in \mathbb{N}$.

Estimates of parameters Q and R can be obtained directly from the covariance matrix for Z_1, Z_2, ..., Z_n, $n \in \mathbb{N}$:

$$\begin{aligned} \hat{Q}_n &= \frac{\sum_{k=1}^{n} z_k^2}{n} + 2\frac{\sum_{k=1}^{n-1} z_k z_{k+1}}{n-1}, \\ \hat{R}_n &= -\frac{\sum_{k=1}^{n-1} z_k z_{k+1}}{n-1}, \end{aligned} \tag{2.3}$$

here z_1, z_2, ..., z_n are particular realizations of Z_1, Z_2, ..., Z_n, respectively.

Formulae (2.3) are suitable not only for the offline but also online estimation as they can be readily calculated recursively. On the other hand, these formulae do not come without their flaws either: 1) estimates $\hat{Q}_n$ and $\hat{R}_n$ might obtain negative values while true parameter values are never meant to be negative (nevertheless, this does not constitute a problem in practical use); 2) estimates $\hat{Q}_n$ and $\hat{R}_n$ approach the true parameter values only asymptotically. Hence, even though there exists a straightforward way to solve the problem under consideration, it is still reasonable to search for more robust solution opportunities that would offer not only an efficient solver for the relatively simplistic model that is being considered in this work, but also provide insight into the solution of similar online identification problems given more complicated models.

This work specifically focuses on the adaptation of the maximum likelihood method for the online solution of the problem described in Section 2.1. The maximum likelihood method is one of the most commonly used methods for drawing statistical inference, and the maximum likelihood estimates have important asymptotic properties, e.g. consistency, functional invariance, and efficiency. Estimates of unknown parameters are obtained by this method as parameters that maximize multivariate probability density. Refer to [3] for more information.

Let us define vector $\mathbf{Z}_n := \begin{pmatrix} Z_1, & Z_2, & ..., & Z_n \end{pmatrix}^\top$. It is straightforward to see that vector $\mathbf{Z}_n$ has n–dimensional Gaussian distribution and therefore its probability density is

$$f_{\mathbf{Z}_n}(z_1,z_2,...,z_n) = \frac{e^{-\frac{1}{2}\mathbf{z}_n^\top \boldsymbol{\Sigma}_n^{-1}\mathbf{z}_n}}{(2\pi)^{\frac{n}{2}}|\boldsymbol{\Sigma}_\mathbf{n}|^{\frac{1}{2}}}, \tag{2.4}$$

here $\mathbf{z}_n := \begin{pmatrix} z_1, & z_2, & ..., & z_n \end{pmatrix}^\top$ and $\boldsymbol{\Sigma}_\mathbf{n} := [\mathrm{Cov}(Z_k,Z_l); 1 \le k,l \le n]$.

By taking the logarithm of (2.4), we get the logarithmic likelihood function:

$$\mathcal{L}_n(\theta) := \mathcal{L}_n(\theta; \mathbf{z}_n) = -\frac{1}{2}\left(\ln(|\mathbf{\Sigma_n}|) + \mathbf{z}_n^\top \mathbf{\Sigma}_n^{-1}\mathbf{z_n} + n\ln(2\pi)\right).$$

Since it is true that

$$\arg\max_\theta \left[\mathcal{L}_n(\theta)\right] = \arg\max_\theta \left[\frac{\mathcal{L}_n(\theta)}{n}\right] = \arg\min_\theta \left[\frac{\ln(|\mathbf{\Sigma_n}|) + \mathbf{z_n}^\top \mathbf{\Sigma_n}^{-1}\mathbf{z_n}}{n}\right],$$

for both normalization and simplicity purposes we will further consider the minimization of the function $\mathcal{L}_n(\theta)$ having the following expression:

$$\mathcal{L}_n(\theta) := \frac{\ln(|\mathbf{\Sigma}_n|) + \mathbf{z}_n^\top \mathbf{\Sigma}_n^{-1}\mathbf{z}_n}{n}. \tag{2.5}$$

Let us now introduce the following parameterization of θ:

$$\begin{aligned} q &:= Q + 2R + \sqrt{(Q+2R)^2 - (2R)^2},\\ r &:= Q + 2R - \sqrt{(Q+2R)^2 - (2R)^2},\\ s &:= \ln\left(\frac{q}{r}\right). \end{aligned} \tag{2.6}$$

We now proceed by writing an estimate for parameter r and constructing a new likelihood function that depends only on parameter s:

Lemma 1. *Function $\hat{r}_n(s)$ such that $\left.\frac{\mathrm{d}L_n(s,r)}{\mathrm{d}r}\right|_{r=\hat{r}_n(s)} = 0$ is a valid estimate of parameter r, here*

$$L_n(s,r) = \frac{1}{n}\ln|\mathbf{A_n}(\mathbf{s})| + \frac{s}{2} + \ln\left(\frac{r}{2}\right) + \frac{2}{r}e^{-\frac{s}{2}}\frac{\mathbf{z_n}^\top\left(\mathbf{A_n}(\mathbf{s})\right)^{-1}\mathbf{z_n}}{n} \tag{2.7}$$

and

$$\hat{r}_n(s) = 2e^{-\frac{s}{2}}\frac{\mathbf{z_n}^\top\left(\mathbf{A_n}(\mathbf{s})\right)^{-1}\mathbf{z_n}}{n}, \tag{2.8}$$

where

$$\mathbf{A_n}(\mathbf{s}) = \left[\begin{cases} 2\cosh\left(\frac{s}{2}\right), & i = j\\ -1, & |i-j| = 1\,; 1 \le i,j \le n\\ 0, & \textit{otherwise}\end{cases}\right].$$

Corollary 1.

$$L_n(s,r) \approx L_n(s,\hat{r}_n(s)) = \frac{1}{n}\ln|\mathbf{A_n}(\mathbf{s})| + 1 + \ln\left(\frac{\mathbf{z_n}^\top\left(\mathbf{A_n}(\mathbf{s})\right)^{-1}\mathbf{z_n}}{n}\right) =: \tilde{L}_n(s). \tag{2.9}$$

The algorithm constructed in this work is based on the minimization of function $\tilde{L}_n(s)$ (see (2.9)) given that $s \in [0,\infty]$. Having found $\hat{s}_n \in [0,\infty]$ with which $\tilde{L}_n(\hat{s}_n) \leq \tilde{L}_n(s)$ for all $s \in [0,\infty]$, $\hat{r}_n$ is then obtained from (2.8). Such $\hat{s}_n$ is either one of all local minima that belong to the interval $(0,\infty)$, i.e. all such points $\bar{s}_n \in (0,\infty)$ for which $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}|_{s=\bar{s}_n} = 0$, $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}|_{s=\bar{s}_n-\epsilon} < 0$ and $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}|_{s=\bar{s}_n+\epsilon} > 0$, here $\epsilon > 0$ is arbitrarily low, or the left-end point of the interval, i.e. 0, or the right-end point of the interval, i.e. ∞. The first derivative of $\tilde{L}_n(s)$, expressed in a specific form to be used by the algorithm, is given in Lemma 2.

Lemma 2. *The first derivate of $\tilde{L}_n(s)$, i.e. $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}$, has the following expressions:*

$$\begin{aligned} \frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s} &= \sinh\left(\frac{s}{2}\right)\left(\mathrm{tr}\left(\frac{(\mathbf{A_n}(\mathbf{s}))^{-1}}{n}\right) - \frac{\mathbf{z_n}^\top (\mathbf{A_n}(\mathbf{s}))^{-2}\mathbf{z_n}}{\mathbf{z_n}^\top (\mathbf{A_n}(\mathbf{s}))^{-1}\mathbf{z_n}}\right) \\ &= \frac{1}{\sinh\left(\frac{s}{2}\right)}\left(\frac{g_n(s)}{h_n(s)} - \frac{1}{2}\left(1+\frac{1}{n}\right)\left[(\mathbf{A_n}(\mathbf{s}))^{-1}\right]_{n,n}\right), \end{aligned} \tag{2.10}$$

here

$$\begin{aligned} h_n(s) &= \frac{1}{n}\frac{\mathbf{z_n}^\top (\mathbf{A_n}(\mathbf{s}))^{-1}\mathbf{z_n}}{\left[(\mathbf{A_n}(\mathbf{s}))^{-1}\right]_{n,n}}, \\ g_n(s) &= \frac{1}{n}\mathbf{z_n}^\top\left(\frac{\frac{1}{2}\cosh\left(\frac{s}{2}\right)(\mathbf{A_n}(\mathbf{s}))^{-1} - \left(\sinh\left(\frac{s}{2}\right)(\mathbf{A_n}(\mathbf{s}))^{-1}\right)^2}{\left[(\mathbf{A_n}(\mathbf{s}))^{-1}\right]_{n,n}}\right)\mathbf{z_n}. \end{aligned} \tag{2.11}$$

The idea of the algorithm is to keep certain parts of (2.10) fixed while in search of the root. If these parts are chosen properly, the newly obtained function will not only become stable in regards of computational complexity but also its root will approach that of the original function, i.e. $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}$, as $n \to \infty$.

In addition to functions $h_n(s)$ and $g_n(s)$ defined by (2.11), we introduce a few similar functions that will be chosen for fixation:

$$\begin{aligned} c_n(s) &= \frac{\mathbf{z_n}^\top \left[(\mathbf{A_n}(\mathbf{s}))^{-1}\right]^{<n>}}{\left[(\mathbf{A_n}(\mathbf{s}))^{-1}\right]_{n,n}}, \\ d_n(s) &= \frac{\mathbf{z_n}^\top \left[(\mathbf{A_n}(\mathbf{s}))^{-2}\right]^{<n>}}{\left[(\mathbf{A_n}(\mathbf{s}))^{-2}\right]_{n,n}}, \end{aligned} \tag{2.12}$$

here $<>$ denotes matrix column.

Since the functions defined by (2.11) and (2.12) involve sums, we must be able to calculate them recursively:

Lemma 3. *For $i = m+1,\ m+2,\ \ldots,\ n$, here $n > m \geq 1$ ($m \in \mathbb{N}$, $n \in \mathbb{N}$), the following* *recursions hold:*

$$
\begin{aligned}
\tilde{h}_i &= a_i^{(1)}(\hat{s}_i)\left(1-\frac{1}{i}\right)\tilde{h}_{i-1} + \frac{1}{i}\left(a_i^{(2)}(\hat{s}_i)\tilde{c}_{i-1}+z_i\right)^2,\\
\tilde{g}_i &= a_i^{(1)}(\hat{s}_i)\left(1-\frac{1}{i}\right)\tilde{g}_{i-1}\\
&\qquad + \frac{1}{i}\Big(a_i^{(3)}(\hat{s}_i)\left(a_i^{(2)}(\hat{s}_i)\tilde{c}_{i-1}+z_i\right)^2 - a_i^{(4)}(\hat{s}_i)\tilde{d}_{i-1}\left(a_i^{(2)}(\hat{s}_i)\tilde{c}_{i-1}+z_i\right)\Big),\\
\tilde{c}_i &= a_i^{(2)}(\hat{s}_i)\tilde{c}_{i-1}+z_i,\\
\tilde{d}_i &= a_i^{(5)}(\hat{s}_i)\tilde{d}_{i-1}+a_i^{(2)}(\hat{s}_i)\tilde{c}_{i-1}+z_i,
\end{aligned}
\tag{2.13}
$$

here

$$
\begin{aligned}
a_i^{(1)}(\hat{s}_i) &= \frac{\left[(\mathbf{A_{i-1}}(\mathbf{\hat{s}_i}))^{-1}\right]_{i-1,i-1}}{\left[(\mathbf{A_i}(\mathbf{\hat{s}_i}))^{-1}\right]_{i,i}} = \frac{\left(1-e^{-\hat{s}_i(i-1)}\right)\left(1-e^{-\hat{s}_i(i+1)}\right)}{\left(1-e^{-\hat{s}_i i}\right)^2},\\
a_i^{(2)}(\hat{s}_i) &= \left[(\mathbf{A_{i-1}}(\mathbf{\hat{s}_i}))^{-1}\right]_{i-1,i-1} = e^{-\frac{\hat{s}_i}{2}}\frac{1-e^{-\hat{s}_i(i-1)}}{1-e^{-\hat{s}_i i}},\\
a_i^{(3)}(\hat{s}_i) &= \sinh\left(\frac{\hat{s}_i}{2}\right)\left(\frac{1}{2}\frac{\cosh\left(\frac{\hat{s}_i}{2}\right)}{\sinh\left(\frac{\hat{s}_i}{2}\right)} - \sinh\left(\frac{\hat{s}_i}{2}\right)\frac{\left[(\mathbf{A_i}(\mathbf{\hat{s}_i}))^{-2}\right]_{i,i}}{\left[(\mathbf{A_i}(\mathbf{\hat{s}_i}))^{-1}\right]_{i,i}}\right)\\
&= \frac{1-e^{-\hat{s}_i}}{2}\left(\frac{e^{-\frac{\hat{s}_i}{2}}}{1-e^{-\hat{s}_i}} + i\frac{e^{-\hat{s}_i\left(i-\frac{1}{2}\right)}}{1-e^{-\hat{s}_i i}} - (i+1)\frac{e^{-\hat{s}_i\left(i+\frac{1}{2}\right)}}{1-e^{-\hat{s}_i(i+1)}}\right),\\
a_i^{(4)}(\hat{s}_i) &= 2\sinh\left(\frac{\hat{s}_i}{2}\right)^2\left[(\mathbf{A_{i-1}}(\mathbf{\hat{s}_i}))^{-2}\right]_{i-1,i-1}\\
&= \frac{1-e^{-\hat{s}_i}}{2}\frac{1-e^{-\hat{s}_i(2i-1)}-(2i-1)\left(1-e^{-\hat{s}_i}\right)e^{-\hat{s}_i(i-1)}}{\left(1-e^{-\hat{s}_i i}\right)^2},\\
a_i^{(5)}(\hat{s}_i) &= \frac{\left[(\mathbf{A_{i-1}}(\mathbf{\hat{s}_i}))^{-2}\right]_{i-1,i-1}}{\left[(\mathbf{A_i}(\mathbf{\hat{s}_i}))^{-2}\right]_{i,i}}\left[(\mathbf{A_i}(\mathbf{\hat{s}_i}))^{-1}\right]_{i,i}\\
&= e^{-\frac{\hat{s}_i}{2}}\frac{1-e^{-\hat{s}_i(i+1)}}{1-e^{-\hat{s}_i i}}\frac{1-e^{-\hat{s}_i(2i-1)}-(2i-1)\left(1-e^{-\hat{s}_i}\right)e^{-\hat{s}_i(i-1)}}{1-e^{-\hat{s}_i(2i+1)}-(2i+1)\left(1-e^{-\hat{s}_i}\right)e^{-\hat{s}_i i}},
\end{aligned}
\tag{2.14}
$$

and $\tilde{h}_m := h_m(\hat{s}_m)$, $\tilde{g}_m := g_m(\hat{s}_m)$, $\tilde{c}_m := c_m(\hat{s}_m)$, $\tilde{d}_m := d_m(\hat{s}_m)$ *(note that if* $\hat{s}_m = \hat{s}_{m+1} = ... = \hat{s}_n = s$, *then* $\tilde{h}_n$, $\tilde{g}_n$, $\tilde{c}_n$, $\tilde{d}_n$ *are respectively equal to* $h_n(s)$, $g_n(s)$, $c_n(s)$, $d_n(s)$ *defined by* (2.11) *and* (2.12)*).*

Having derived recursions (2.13), we can now implement the previously described idea of the algorithm. We replace h_n and g_n in (2.10) with their recursive counterparts $\tilde{h}_n$ and $\tilde{g}_n$. During the root search, the terms $\tilde{h}_{n-1}$, $\tilde{g}_{n-1}$, $\tilde{c}_{n-1}$ and $\tilde{d}_{n-1}$ are now fixed as they do not depend on $\hat{s}_n$. The newly found estimate of parameter s, i.e. $\hat{s}_n$, is then used together with the newest available value of Z, i.e. z_n, to recursively obtain $\tilde{h}_n$, $\tilde{g}_n$, $\tilde{c}_n$ and $\tilde{d}_n$. More specifically, for $i = m+1$, $m+2$, ..., n, here $n > m \geq 1$ ($m \in \mathbb{N}$, $n \in \mathbb{N}$), the algorithm alternates between the following steps:

- **Estimation.** Equation $k_i := k_i(s,h,g,c,d) = 0$ is solved for s, such that minimizes function $K_i := \int k_i ds$ in the interval $[0; \infty]$ (in practice, we only search for a local minimum), here

$$\begin{aligned} k_i = g a_i^{(1)}(s)\left(1-\frac{1}{i}\right)+\frac{1}{i}\left(a_i^{(3)}(s)\left(c a_i^{(2)}(s)+z_i\right)^2-d a_i^{(4)}(s)\left(c a_i^{(2)}(s)+z_i\right)\right) \\ -\frac{1}{2}\left(1+\frac{1}{i}\right) a_{i+1}^{(2)}(s)\left(h a_i^{(1)}(s)\left(1-\frac{1}{i}\right)+\frac{1}{i}\left(c a_i^{(2)}(s)+z_i\right)^2\right) \end{aligned} \tag{2.15}$$

and $s = \hat{s}_i$, $h = \tilde{h}_{i-1}$, $g = \tilde{g}_{i-1}$, $c = \tilde{c}_{i-1}$, $d = \tilde{d}_{i-1}$.

- **Update.** $\tilde{h}_i$, $\tilde{g}_i$, $\tilde{c}_i$, $\tilde{d}_i$ are calculated by (2.13).

As the values of parameter s range from 0 to ∞, we need to calculate the limits at these extreme values for the general function and its derivative, i.e. $\tilde{L}_n(s)$ and $\frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s}$, as well as for the constructed function $k_n(s,h,g,c,d)$ and its constituents $h_n(s)$, $g_n(s)$, $c_n(s)$, and $d_n(s)$. Doing so provides information ranging from useful to necessary for the proposed algorithm.

Lemma 4. *When s approaches zero, the following limits hold:*

$$\begin{aligned}
&(a.1) \quad \lim_{s\to 0} \tilde{L}_n(s) = 1 + \frac{\ln(n+1)}{n} + \ln\left(u_n^{(1)}\right),\\
&(a.2) \quad \lim_{s\to 0} \frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s} = \frac{1}{12}\frac{u_n^{(2)}}{u_n^{(1)}} \lim_{s\to 0} s = 0,\\
&(a.3) \quad \lim_{s\to 0} h_n(s) = \left(1+\frac{1}{n}\right) u_n^{(1)},\\
&(a.4) \quad \lim_{s\to 0} g_n(s) = \frac{1}{2}\lim_{s\to 0} h_n(s),\\
&(a.5) \quad \lim_{s\to 0} c_n(s) = y_n - u_n^{(3)},\\
&(a.6) \quad \lim_{s\to 0} d_n(s) = \sum_{i=1}^{n}\left((y_i - y_0)\frac{3i(i+1)-n(n+2)}{n(2n+1)}\right),\\
&(a.7) \quad \lim_{s\to 0} k_n(s,h,g,c,d) = \left(1-\frac{1}{n^2}\right)\left(1-\frac{1}{n}\right)\left(g-\frac{h}{2}\right),\\
&(a.7') \quad \textit{if } g = \frac{h}{2}, \textit{ then } \lim_{s\to 0}\frac{k_n(s,h,g,c,d)}{s^2} =\\
&\qquad = \frac{1}{12} a_n^{(2,0)}\left(h\frac{(2n+1)\left(n^2-1\right)}{(2n)^2} - d\left(1+a_n^{(2,0)}\right)\left(c a_n^{(2,0)}+z_n\right) - \frac{1}{2}\left(c a_n^{(2,0)}+z_n\right)^2\right),
\end{aligned} \tag{2.16}$$

here

$$
\begin{aligned}
u_n^{(1)} &= \frac{\sum_{i=1}^{n}\left[\frac{i}{i+1}\left(y_i - \frac{\sum_{j=0}^{i-1} y_j}{i}\right)^2\right]}{n}, \\
u_n^{(2)} &= \sum_{j=0}^{n-1}\sum_{i=0}^{n-1}\left[z_{i+1}z_{j+1}\left(\frac{1+\min(i,j)}{n+1}\right)\left(1-\frac{\max(i,j)}{n}\right)\right. \\
&\qquad\left.\left(j^2+i^2+2\min(i,j)+1-n\left(2\max(i,j)+1\right)\right)\right], \\
u_n^{(3)} &= \frac{\sum_{i=0}^{n-1} y_i}{n}, \\
a_n^{(2,0)} &= 1-\frac{1}{n},
\end{aligned}
\tag{2.17}
$$

and y_0, y_1, ..., y_n are particular realizations of Y_0, Y_1, ...,Y_n, respectively ($n \in \mathbb{N}$).

When s approaches infinity, the following limits hold:

$$
\begin{aligned}
&(b.1) \quad \lim_{s\to\infty} \tilde{L}_n(s) = 1+\ln\left(v_n^{(1)}\right), \\
&(b.2) \quad \lim_{s\to\infty} \frac{\mathrm{d}\tilde{L}_n(s)}{\mathrm{d}s} = -\frac{v_n^{(2)}}{v_n^{(1)}}\lim_{s\to\infty} e^{-\frac{s}{2}} = 0, \\
&(b.3) \quad \lim_{s\to\infty} h_n(s) = v_n^{(1)}, \\
&(b.4) \quad \lim_{s\to\infty} g_n(s) = -\frac{1}{2}v_n^{(2)}, \\
&(b.5) \quad \lim_{s\to\infty} c_n(s) = z_n, \\
&(b.6) \quad \lim_{s\to\infty} d_n(s) = z_n, \\
&(b.7) \quad \lim_{s\to\infty} k_n(s,h,g,c,d) = g\left(1-\frac{1}{n}\right) - d\frac{z_n}{2n},
\end{aligned}
\tag{2.18}
$$

here

$$
\begin{aligned}
v_n^{(1)} &= \frac{\sum_{i=1}^{n}(z_i)^2}{n}, \\
v_n^{(2)} &= \frac{\sum_{i=1}^{n-1} z_i z_{i+1}}{n}.
\end{aligned}
\tag{2.19}
$$

The limits of $h_n(s)$, $g_n(s)$, $c_n(s)$, and $d_n(s)$ when s approaches its extreme values can be readily tracked at every iteration. This is rather useful as it allows beginning with a fresh sequence of s estimates in case its estimate becomes zero (in case it becomes infinity, the same result is achieved with recursive updates). Moreover, the values at the limiting values of s can be also tracked for the general function (2.9) and its derivative (2.10), hence providing an opportunity to correctly estimate s in advance as zero or infinity with some probability. For all this, the following lemma is needed:

Lemma 5. *The following recursive relations hold:*

$$\begin{aligned}
&(1)\quad u_n^{(1)} = u_{n-1}^{(1)}\left(1-\frac{1}{n}\right) + \frac{1}{n+1}\left(\lim_{s\to 0} c_n(s)\right)^2,\\
&(2)\quad u_n^{(2)} = u_{n-1}^{(2)} ...,\\
&(3)\quad u_n^{(3)} = u_{n-1}^{(3)} + \frac{y_{n-1} - u_{n-1}^{(3)}}{n},\\
&(4)\quad v_n^{(1)} = v_{n-1}^{(1)} + \frac{(z_n)^2 - v_{n-1}^{(1)}}{n},\\
&(5)\quad v_n^{(2)} = v_{n-1}^{(2)} + \frac{z_{n-1}z_n - v_{n-1}^{(2)}}{n},\\
&(6)\quad \lim_{s\to 0} d_n(s) = \left(1-\frac{1}{n}\right)\frac{2n-1}{2n+1}\lim_{s\to 0} d_{n-1}(s) + \lim_{s\to 0} c_n(s).
\end{aligned} \tag{2.20}$$

Having estimated parameter s at the sample size n, the estimate for parameter r can be retrieved from (2.8) using the recursively calculated term $\tilde{h}_n$:

$$\hat{r}_n(\hat{s}_n,\tilde{h}_n) = 2e^{-\frac{\hat{s}_n}{2}} a_{n+1}^{(2)}(\hat{s}_n)\tilde{h}_n. \tag{2.21}$$

Finally, when the estimates of parameters s and t are known, the estimates of unknown model variances Q and R are obtained by the following formulae:

$$\begin{aligned}
\hat{Q}_n(\hat{s}_n,\hat{r}_n) &= \frac{\hat{r}_n\left(e^{\frac{\hat{s}_n}{2}}-1\right)^2}{2},\\
\hat{R}_n(\hat{s}_n,\hat{r}_n) &= \frac{\hat{r}_n e^{\frac{\hat{s}_n}{2}}}{2}.
\end{aligned} \tag{2.22}$$

However, since $lim_{\hat{s}_n\to\infty}\hat{r}_n(\hat{s}_n,\tilde{h}_n) = 0$, estimates (2.22) are not valid when $\hat{s}_n$ is equal to ∞. In this case, the estimates of Q and R are the following:

$$\begin{aligned}
&\hat{Q}_n = v_n^{(1)} \quad \text{(follows from (2.3))},\\
&\hat{R}_n = 0 \quad \text{(by the definition of } s\text{)}.
\end{aligned} \tag{2.23}$$

In the similar regard, when $\hat{s}_n = 0$, we have:

$$\begin{aligned}
\hat{Q}_n &= 0,\\
\hat{R}_n &= \left(1-\frac{1}{n^2}\right)u_n^{(1)}.
\end{aligned} \tag{2.24}$$

In the next sub-subsection, we lay out the full algorithm.

2.2.2 Algorithm

1 algoritmas Incremental maximum likelihood estimation of unknown model parameters Q and R

Input: A sequence of z's (y's) generated with model variances Q and R, i.e. z_1, z_2, ..., z_m, ..., z_n, here m is the initial sample size and n is the total sample size (m, $n \in \mathbb{N}$, $n > m$).

Output: Estimates of Q and R at different sample sizes (iterations), i.e. $\hat{Q}_i$ and $\hat{R}_i$, here $i = m$, $m+1$, ..., n.

$u^{(1)} \leftarrow u_m^{(1)}$, $u^{(2)} \leftarrow u_m^{(2)}$, $u^{(3)} \leftarrow u_m^{(3)}$ (see (2.17)); $v^{(1)} \leftarrow v_m^{(1)}$, $v^{(2)} \leftarrow v_m^{(2)}$ (see (2.19)); $d_{lim_0} \leftarrow \lim_{s \to 0} d_m(s)$ (see (2.16))
$\hat{s} \leftarrow \min\limits_{s} \tilde{L}_m(s)$ (see (2.9))
if $\hat{s} = \infty$ **then**
    $h \leftarrow v^{(1)}$, $g \leftarrow -\frac{1}{2} v^{(2)}$, $c \leftarrow z_m$, $d \leftarrow z_m$
    $\hat{Q}_m \leftarrow v^{(1)}$, $\hat{R}_m \leftarrow 0$
else if $\hat{s} = 0$ **then**
    $h \leftarrow \left(1 + \frac{1}{m}\right) u^{(1)}$, $g \leftarrow \frac{1}{2} h$, $c \leftarrow y_m - u^{(3)}$, $d \leftarrow d_{lim_0}$
    $\hat{Q}_m \leftarrow 0$, $\hat{R}_m \leftarrow \left(1 - \frac{1}{m^2}\right) u^{(1)}$
else
    $h \leftarrow h_m(\hat{s})$, $g \leftarrow g_m(\hat{s})$ (see (2.11)); $c \leftarrow c_m(\hat{s})$, $d \leftarrow d_m(\hat{s})$ (see (2.12))
    $\hat{r} \leftarrow \hat{r}_m(\hat{s}, h)$ (see (2.21))
    $\hat{Q}_m \leftarrow \hat{Q}_m(\hat{s}, \hat{r})$, $\hat{R}_m \leftarrow \hat{R}_m(\hat{s}, \hat{r})$ (see (2.22))
end if
for $i = m+1$, $m+2$, ..., n **do**
    $u^{(1)} \leftarrow u_i^{(1)}$, $u^{(2)} \leftarrow u_i^{(2)}$, $u^{(3)} \leftarrow u_i^{(3)}$, $v^{(1)} \leftarrow v_i^{(1)}$, $v^{(2)} \leftarrow v_i^{(2)}$, $d_{lim_0} \leftarrow \lim_{s \to 0} d_i(s)$, where the right-hand sides are calculated recursively by (2.20)
    $L^{(0)} \leftarrow 1 + \frac{\ln(i+1)}{i} + \ln\left(u^{(1)}\right)$, $L^{(\infty)} \leftarrow 1 + \ln\left(v^{(1)}\right)$
    $l^{(0)} \leftarrow \frac{u^{(2)}}{u^{(1)}}$, $l^{(\infty)} \leftarrow -\frac{v^{(2)}}{v^{(1)}}$
    $k^{(\infty)} \leftarrow g\left(1 - \frac{1}{i}\right) - d\frac{z_i}{2i}$, $k^{(0)} \leftarrow \left(1 - \frac{1}{i^2}\right)\left(1 - \frac{1}{i}\right)\left(g - \frac{h}{2}\right)$, $k^{(-1)} \leftarrow \lim_{s \to 0} \frac{k_i(s,h,g,c,d)}{s^2}$ (see (2.16))
    if $\left(l^{(0)} < 0\right.$ **and** $\left.l^{(\infty)} < 0\right)$ **or** $\left(l^{(0)} > 0\right.$ **and** $l^{(\infty)} < 0$ **and** $\left.L^{(\infty)} < L^{(0)}\right)$ **or** $\left(k^{(0)} < 0\right.$ **and** $\left.k^{(\infty)} < 0\right)$ **or** $\left(k^{(0)} = 0\right.$ **and** $k^{(-1)} < 0$ **and** $\left.k^{(\infty)} < 0\right)$ **then**
        $\hat{s} \leftarrow \infty$
        $h \leftarrow v^{(1)}$, $g \leftarrow -\frac{1}{2} v^{(2)}$, $c \leftarrow z_i$, $d \leftarrow z_i$
        $\hat{Q}_i \leftarrow v^{(1)}$, $\hat{R}_i \leftarrow 0$

else if $\left(l^{(0)} > 0 \ \textbf{and} \ l^{(\infty)} > 0\right)$ **or** $\left(l^{(0)} > 0 \ \textbf{and} \ l^{(\infty)} < 0 \ \textbf{and} \ L^{(0)} < L^{(\infty)}\right)$ **or** $\left(k^{(0)} > 0 \ \textbf{and} \ k^{(\infty)} > 0\right)$ **or** $\left(k^{(0)} = 0 \ \textbf{and} \ k^{(-1)} > 0 \ \textbf{and} \ k^{(\infty)} > 0\right)$ **then**

$\hat{s} \leftarrow 0$

$h \leftarrow \left(1 + \frac{1}{i}\right) u^{(1)}$, $g \leftarrow \frac{1}{2}h$, $c \leftarrow y_i - u^{(3)}$, $d \leftarrow d_{limo}$

$\hat{Q}_i \leftarrow 0$, $\hat{R}_i \leftarrow \left(1 - \frac{1}{i^2}\right) u^{(1)}$

else

Starting from $\hat{s}$ (if $\hat{s} \neq 0$ and $\hat{s} \neq \infty$) or some arbitrary low value (e.g. 1) not equal neither to zero nor infinity (if $\hat{s} = 0$ or $\hat{s} = \infty$), bracket the interval in which there exists the root of k_i (see (2.15)) corresponding to the local minumum

Find the root of k_i (see (2.15)) in the bracketed interval. Assign this root to $\hat{s}$

$h \leftarrow \tilde{h}_i$, $g \leftarrow \tilde{g}_i$, $c \leftarrow \tilde{c}_i$, $d \leftarrow \tilde{d}_i$, here the right-hand sides are calculated recursively by (2.13) using $\hat{s}$

$\hat{r} \leftarrow \hat{r}_i(\hat{s},h)$ (see (2.21))

$\hat{Q}_i \leftarrow \hat{Q}_i(\hat{s},\hat{r})$, $\hat{R}_i \leftarrow \hat{R}_i(\hat{s},\hat{r})$ (see (2.22))

end if

end for

In the next subsection, we test the proposed algorithm experimentally.

2.3 Experimental Results

Algorithm 1 was tested experimentally by a Monte Carlo method [4]. Three different experiments were carried out, each consisting of 100 runs of the algorithm with $m = 10$, $n = 10^5$, and varying values of parameters Q and R. The results are depicted in Figures 1-6 (Figures 1-2, 3-4 and 5-6 represent the first, second and third experiments, respectively).

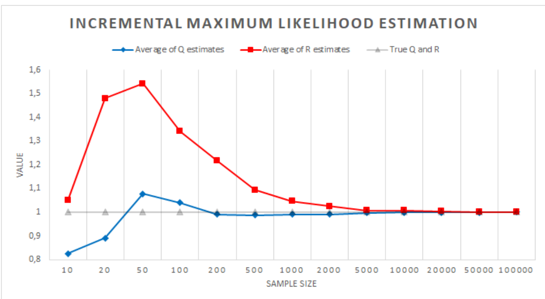

Figure 1: 100 runs of Algorithm 1 with $Q = 1$, $R = 1$, $m = 10$, $n = 10^5$. Averages are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

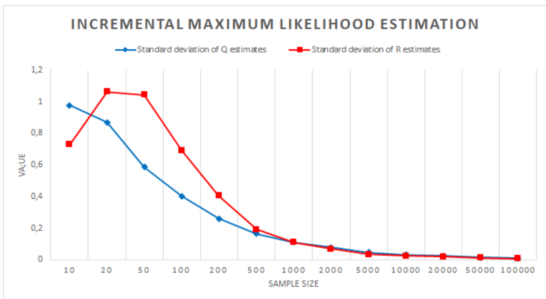

Figure 2: The same runs of Algorithm 1 as in Figure 1. Standard deviations are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

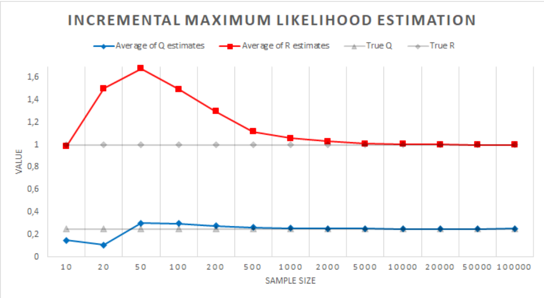

Figure 3: 100 runs of Algorithm 1 with $Q = 0.25$, $R = 1$, $m = 10$, $n = 10^5$. Averages are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

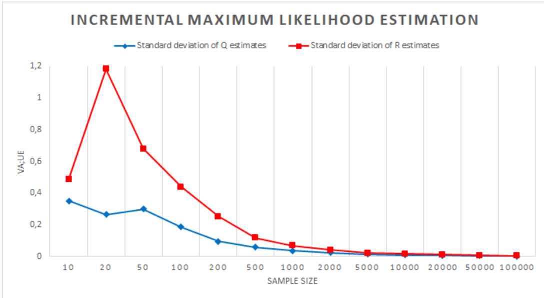

Figure 4: The same runs of Algorithm 1 as in Figure 3. Standard deviations are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

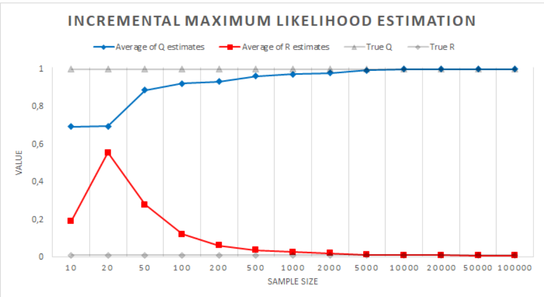

Figure 5: 100 runs of Algorithm 1 with $Q = 1$, $R = 0.01$, $m = 10$, $n = 10^5$. Averages are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

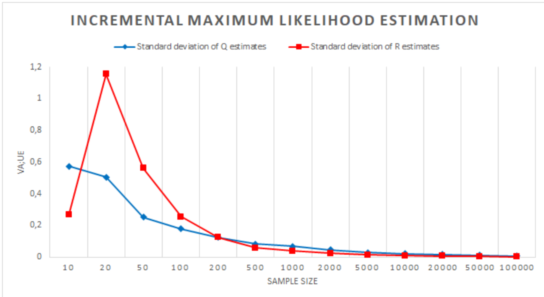

Figure 6: The same runs of Algorithm 1 as in Figure 5. Standard deviations are plotted against a sample size of 10, 20, 50, 10^2, $2 \cdot 10^2$, $5 \cdot 10^2$, 10^3, $2 \cdot 10^3$, $5 \cdot 10^3$, 10^4, $2 \cdot 10^4$, $5 \cdot 10^4$, 10^5.

The fact that the average estimates of both Q and R tend to their true values and the standard deviations of both Q and R tend to zero (as the sample size increases) verifies the convergence of the proposed algorithm.

3 Modelling the Impact of Cultural Participation on Social Capital

This section considers the development of models for the impact of cultural participation on social capital [5]. We assume an abstract space consisting of a set of actors representing individual human beings and a set of cultural events representing different types of cultural events. The goal is to model the social impact (in terms of social capital, the concept of which we do not detail here) on actors deriving from cultural participation subject to the stationary flow of independent events (SFIE).

3.1 Probabilities of Cultural Participation

We introduce the frequencies of actors' profiles, probabilities of cultural events and probabilities of actors' participation in those cultural events.

First, assuming that K different actors' profiles are ascertained, then the probability of a randomly selected actor possessing the k^{th} profile is q_k, $1 \leq k \leq K$, here $\sum_{k=1}^{K} q_k = 1$.

Second, we consider a set of cultural events consisting of m different types of events. We denote the indicator of participation of some k^{th} profile actor in the event of i^{th} type as χ_i^k. Thus, $\chi_i^k = 1$ if the participation took place, and $\chi_i^k = 0$ if it did not. We assume, for simplicity, that one actor can only participate in one event at a given time. Thus, $\chi_i^k \cdot \chi_j^k = 0$ if $i \neq j$. Then, conditional probabilities of participation in cultural events make up the *matrix of probabilities of preferences*:

$$Pr\left(\chi_i^k = 1 | i, k\right) = P_i^k, \tag{3.1}$$

here $1 \leq k \leq K$, $1 \leq i \leq m$.

Given that different types of events in the set of cultural events are numbered from 1 to *m*, the flow of cultural events is presented as a sequence of numbers from 1 to m. Thus, events in time are considered as discrete or continuous time sequences. The SFIE of m types can be described by a vector of *event probabilities* $\pi = \{\pi_i\}_{i=1}^{m}$, here $\sum_{i=1}^{m} \pi_i = 1$. Hence, the probability of the k^{th} profile actor to participate in the event of i^{th} type following from some SFIE is expressed using conditional probabilities (1) in the following way:

$$Pr\left(\chi_i^k = 1 | k\right) = p_i^k = \pi_i \cdot P_i^k, \tag{3.2}$$

here $1 \leq k \leq K$, $1 \leq i \leq m$. Conversely, the probability of the k^{th} profile actor not participating is

$$Pr\left(\sum_{i=1}^{m} \chi_i^k = 0 | k\right) = 1 - \sum_{i=1}^{m} \pi_i \cdot P_i^k. \tag{3.3}$$

3.2 Finite-Difference Model of the Social Impact of Cultural Participation

Let us develop the model for the simulation of the social impact of participation in cultural events flow. We consider the social impact of the dynamics of cultural processes as a systematic process that is realised through a *change of social capital* in conjunction with the *flow of cultural events* and the *population of actors*. We assume that social capital is measured by various social capital indexes, the set of those is denoted by Θ. The values of these indexes range from 0 to a, here $a > 0$. The total social capital for a single actor, C, can then be measured as the sum of all social capital indexes: $C = \sum_{\theta \in \Theta} C^\theta$.

Each actor at a certain moment in time is distinguished by a certain numerical value C^θ of some social capital index before the event and a certain value C_Δ^θ afterwards (assumption). Although the upper limit of the value of a social capital index is a, taking into account naturally existing differences between people, we can consider that potential social capital (PSC) is individually distributed and, thus, denote it for a certain actor by A^θ, $0 < A^\theta \leq a$, $\theta \in \Theta$. Then, important information is provided by the *social capital development ratio* (SCDR), which describes how an actor is able to assimilate its PSC:

$$D^{\theta} = \frac{C^{\theta}}{A^{\theta} - C^{\theta}}, \tag{3.4}$$

here $0 < C^{\theta} < A^{\theta}$, $\theta \in \Theta$. Due to the non-linearity and limitedness of social capital (assumption), changes of social capital indexes are described more adequately in a multiplicative way using the SCDR, unlike economic-financial capital, whose changes are measured in the usual additive way. According to the model, actors use a randomised strategy for participating in cultural events as follows. We assume that w_0^{θ} and b^{θ} are certain constants; w_0^{θ} describes the change of social capital if there is no actor involvement in the event, and b^{θ} is the expected impact of the event on social capital, $\theta \in \Theta$. Hence, we denote the fact of actor participation in some cultural event taking place in the community during time unit Δt at a certain moment in time t by $\chi = 1$ and no participation by $\chi = 0$. The change of SCDR should depend on the fact of actor participation in the event. Thus, in the simplest case, one can consider that this ratio, having value D^{θ} before the event, is changed to value $D_{\Delta}^{\theta} = D^{\theta} + \Delta D^{\theta}$ after the event in the following way:

$$D_{\Delta}^{\theta} = \begin{cases} D^{\theta} \cdot \left(1 + (w_0^{\theta} + \xi^{\theta}) \cdot \Delta t\right) & if\ \chi = 0 \\ D^{\theta} \cdot \left(1 + (b^{\theta} + \xi^{\theta}) \cdot \Delta t\right) & if\ \chi = 1 \end{cases}, \tag{3.5}$$

where ξ^{θ} represents the overall impact of other factors (assumption), $\theta \in \Theta$. Since the impact is analysed through a sufficiently large number of events, it is assumed that the impact of one separate event is small, i.e. $\left|w_0^{\theta} \cdot \Delta t\right| << 1$, $\left|b^{\theta} \cdot \Delta t\right| << 1$, $\theta \in \Theta$. The fact that participation in the event is expected to change social capital positively and no participation is expected to change it negatively might be modelled by assigning certain values: $w_0^{\theta} \leq 0$, $b^{\theta} > 0$, $\theta \in \Theta$. Due to the complex nature of the overall other factors affecting social capital, it is reasonable to consider their entire impact $\xi = (\xi^{first\_index}, \xi^{second\_index}, \dots, \xi^{last\_index})$ distributed with respect to the Gaussian law $N(\varepsilon,\ \tau)$, where ε and τ are a mean vector and a covariance matrix, respectively. The appropriate choice of ε and τ enables to reflect the social impact of cultural participation in comparison with other factors.

We denote a vector $\chi = \{\chi_i\}_{i=1}^{m}$, where χ_i is an indicator of participation in the event of i^{th} type, $\chi_i \in \{0; 1\}$, $0 \leq \sum_{i=1}^{m} \chi_i \leq 1$, and a vector of cultural event weights $w^{\theta} = \left\{w_i^{\theta}\right\}_{i=1}^{m}$, where $w_i^{\theta} = b_i^{\theta} - w_0^{\theta}$, and b_i^{θ} is the expected impact of an event of i^{th} type, $1 \leq i \leq m$, $\theta \in \Theta$. Eq. (3.5) can be rewritten as the following equation expressing the change of a certain SCDR after the event has occured at a certain time moment:

$$D_{\Delta}^{\theta} = D^{\theta} \cdot \left(1 + \left(w_0^{\theta} + \chi^{T} \cdot w^{\theta} + \xi^{\theta}\right) \cdot \Delta t\right), \tag{3.6}$$

here $\theta \in \Theta$. It should be noted that Eq. (3.6) follows the logistic regression model used for modelling the social impact of culture [6, 7].

Remark. We denote $\Delta\, ln(D) = ln(D_{\Delta}) - ln(D)$. Then,

$$\Delta\, ln(D) \approx \frac{\Delta D}{D} = (w_0 + \chi^{T} \cdot w + \xi) \cdot \Delta t, \tag{3.7}$$

where the upper index is omitted for simplicity.

Using Taylor's formula:

$$\Delta\, ln(D) = ln(D_{\Delta}) - ln(D) = ln\left(1 + \frac{\Delta D}{D}\right) = \frac{\Delta D}{D} + o\left(\frac{\Delta D}{D}\right).$$

Now (3.7) follows because of (3.6).

We can rewrite the relationship between the SCDR and social capital indexes (3.4) as follows:

$$C^{\theta} = A^{\theta} \cdot \frac{D^{\theta}}{1+D^{\theta}}, \quad (3.8)$$

here $\theta \in \Theta$.

Thus, formulae (3.6) and (3.8) represent the finite-difference model of the impact of participation in cultural events on the social capital of actors. According to this model, each actor is characterised by parameters A^{θ} of its PSC, vectors of weighting parameters w^{θ} that measure the effect of participation in cultural events, and parameters $\varepsilon^{\theta},\ \tau^{\theta,\theta}$ of normally distributed ‘noise’ representing other factors that have an impact on social capital ($\theta \in \Theta$).

3.3 Model of the Social Impact of Cultural Participation as a Stochastic Differential Equation

Here we consider a mathematical model that can reveal many properties of the mechanism behind the social impact of culture. We assume that N actors are influenced by the SFIE consisting of m types of events spread over time period T and that the probabilities of event occurrences and actor participation are described by Eq. (3.1), (3.2), (3.3). For simplicity, we assume that the time interval T is a conjunction of discrete time units Δt, during which only one event from the field of cultural events takes place. If the unit of time Δt decreases, then the number of events $M = \frac{T}{\Delta t}$ occurring during the fixed period of time T increases. It should be noted that

$$\chi_i \cdot \chi_j = \begin{cases} \chi_i, & if\ i = j \\ 0, & if\ i \neq j \end{cases}. \quad (3.9)$$

Moreover, according to Eq. (3.2), the probability of the k^{th} profile actor participating in the event of i^{th} type is $E\chi_i = p_i^k = \pi_i \cdot P_i^k$, $1 \le k \le K$, $1 \le i \le m$.

To consider the social impact of a cultural events flow that occurred during some time interval $[0,\ T]$, we denote the total effect of a set of cultural events on individual actors by $\sum(w_0 + \chi^T \cdot w + \xi) \cdot \Delta t$, where notations of social capital indexes and the numbers of events are omitted for simplicity. Then, from the central limit theorem (see, e.g., [8]), it follows that the impact of this flow is distributed with respect to the normal law $N(\mu,\ \sigma^2)$, where the mean is

$$\mu = (w_0 + p^T \cdot w + \varepsilon) \cdot T, \quad (3.10)$$

and the variance is

$$\sigma^2 = (\tau + p^T \cdot w \cdot w^T - (p^T \cdot w) \cdot w^T p) \cdot T, \quad (3.11)$$

here $p = \{p_i^k\}_{i=1,k=1}^{m,K}$. It should be noted that the social capital indexes are correlated.

Eq. (3.4) and (3.7) yield $\Delta\, ln(D) \approx \frac{\Delta C}{C \cdot \left(1 - \frac{C}{A}\right)}$, which implies the next proposition.

Proposition. Let N actors be influenced during time period $[0,\ T]$ by the SFIE consisting of m types of events. We assume that during time unit Δt, only one event takes place, whose particular impact is small, namely $|w_i \cdot \Delta t| << 1$, where w_i, $0 \le i \le m$, are certain weights of event impact, and that the probabilities of event occurrences and participation in the events are given by Eq. (3.1), (3.2), and (3.3). If the time unit Δt decreases so that the number of events during the period increases but the probabilities (3.1), (3.2), and (3.3) remain the same, then the social capital indexes of each of N individual actors follow the system of stochastic differential equations:

$$dC^{\theta} = C^{\theta} \cdot \left(1 - \frac{C^{\theta}}{A^{\theta}}\right) \cdot \left(\mu^{\theta} \cdot dt + dW_t^{\theta}\right), \quad (3.12)$$

here W_t^{θ} is a Wiener process, having a zero mean and a covariance matrix computed using Eq. (3.11), $\theta \in \Theta$.

These equations represent the model with a probabilistic distribution and other abstract properties of social capital indexes, which can be particularly useful for studies of social impact and interactions of multicultural event flows. The equations are derived using assumptions and properties of a rather universal kind that also have applications for other modelling tasks and the simulation of social-behavioural phenomena (see [9]).

4 Apibendrinimas

Ataskaitoje pristatytas realaus laiko identifikavimo algoritmas, skirtas užtriukšmintam Gauso atsitiktinio klaidžiojimo modeliui. Algoritmas tiesiogiai remiasi didžiausio tikėtinumo metodu, panaudodamas atitinkamai parametrizuotos tikėtinumo funkcijos išvestinę. Ieškant jos šaknies, dalis narių yra laikomi konstantomis (gaunama alternatyvi funkcija), kas, didėjant stebėjimų skaičiui, leidžia išlaikyti pastovų skaičiavimų sudėtingumą. Tuo tarpu tinkamai konstantomis parinkti nariai, kurie yra rekursiškai perskaičiuojami naudojant naujausią parametrų įvertį bei užtriukšmintą stebėjimą, užtikrina, kad nėra prarandamas konvergavimas.

Taip pat ataskaitoje pažvelgta į kultūros socialinio poveikio modeliavimą, pateikiant tam skirtą galimą modelį – kandidatą. Nagrinėjant tokius modelius kaip pateiktasis atsiranda filtravimo (prognozavimo) bei identifikavimo uždavinių poreikis potencialiai užtriukšmintų stebėjimų kontekste, kas nagrinėjamą sritį padaro tinkama teorinių rezultatų (algoritmų) taikymams.

Ataskaitos turinys yra dalis esamų ir/ar būsimų straipsnių medžiagos.

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